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COMMENT

A note on current algebra in QCD$_2$ and two-dimensional chiral theories

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Abstract. Current algebra for QCD$_2$ and two-dimensional chiral gauge theories is derived in a straightforward way from the effective action.

In memory of Marysia

Much progress has been made recently in understanding the structure of two-dimensional gauge theories. The main contribution was the explicit evaluation of the fermionic determinant in two dimensions (D'Adda *et al* 1983, Polyakov and Wiegman 1983, Alvarez 1984, Bothelho and Monteiro 1984, Rothe 1986, Nepomechie 1984). Using this result, Witten's bosonisation rules (Witten 1984) were generalised to include the gauge fields (DiVecchia and Rossi 1984, DiVecchia *et al* 1984, Gonzales and Redlich 1984, Gamboa Saravi *et al* 1985a). The bosonisation rules for the fermionic currents allowed in turn the derivation of the current algebra in QCD$_2$ (Gamboa Saravi *et al* 1985b).

The above results were obtained within the Euclidean approach which also has some disadvantages. For example, the current algebra for QCD$_2$ was obtained somewhat indirectly: the commutation rules were written down which reproduced the transformation properties following from the bosonised form of fermionic currents (Gamboa Saravi *et al* 1985b).

On the other hand, we sometimes want to keep the gauge fields unquantised and understand the physics behind this approximation. This is especially the case for anomalous theories where no unique consistent quantisation scheme for the gauge fields seems to exist (at least at the level of renormalisable theories in four dimensions).

While no direct connection between the Minkowskian and Euclidean space versions of external gauge field problem exists, Makowka and Wanders (1985, 1986) showed in an interesting paper how to generalise the results of D'Adda *et al* (1983) and others to the Minkowskian case. Using their results, we rederive here the current algebra for QCD$_2$ as well as for anomalous chiral theories in two dimensions.

Let us start with a short recapitulation of the paper by Makowka and Wanders (1986). The effective action $W[A]=-\mathrm{i}\ln(\Omega_{\mathrm{out}},\Omega_{\mathrm{in}})_A$ for QCD$_2$ with the gauge fields kept unquantised is

$$W[A]=K_+[T_+]+K_-[T_-]+\frac{g^2}{4\pi}\int \mathrm{d}^2x\ \mathrm{Tr}(A_+A_-)(x)$$

$$K_\pm[T]=\frac{1}{4\pi}\int_0^1 \mathrm{d}\tau\int \mathrm{d}^2x\ \mathrm{Tr}[\partial_\pm TT^{-1}\partial_\tau(\partial_\mp TT^{-1})](x,\tau)$$

where $\pm$ mean the light-cone coordinates and the matrices $T_\pm$ satisfy

$$\partial_\mp T_\pm(x,\tau) = \mathrm{i}gA_\mp(x,\tau)T_\pm(x,\tau)$$

together with the Feynman–Stueckelberg boundary conditions. $A_\mu(x,\tau)$ is any interpolation such that $A_\mu(x,0)=0$, $A_\mu(x,1)=A_\mu(x)$ (the compactness of the support of $A_\mu(x,\tau)$ in x for any τ is assumed). In deriving this result, the compactification of spacetime is replaced by the vanishing of boundary terms due to the causality properties of $T_\pm$.

Now, let us define the current

$$\langle j^a_\pm(x)\rangle_A \equiv (\Omega_{\mathrm{out}}, j^a_\pm(x)\Omega_{\mathrm{in}})_A = \frac{-2\mathrm{i}}{g}\frac{\delta}{\delta A^a_\mp(x)}[\exp(\mathrm{i}W[A])] \tag{1}$$

or, explicitly (Makowka and Wanders 1986),

$$\langle j^a_\pm(x)\rangle_A = \left[\frac{\mathrm{i}}{2\pi}\mathrm{Tr}(\partial_\pm T_\pm(x)T^{-1}_\pm(x)\lambda_a) + \frac{g}{4\pi}A^a_\pm(x)\right]\exp(\mathrm{i}W[A]). \tag{2}$$

One can easily check that $j^a_\mu(x)$ is conserved, $\langle D_+j_- + D_-j_+\rangle = 0$, and transforms covariantly under the gauge transformations of the background field, $\delta\langle j^a_\mu(x)\rangle_A = \varepsilon^b(x)f_{bac}\langle j^c_\mu(x)\rangle_A$.

To derive the current algebra we calculate the two-point functions. We obtain (using the properties of $T_\pm$ listed in Makowka and Wanders (1986))

$$\exp(-\mathrm{i}W[A])\langle T^a_\pm(x)j^b_\pm(y)\rangle_c = \frac{\mathrm{i}}{\pi}\frac{\partial D_\pm(x-y)}{\partial x^\mp}\mathrm{Tr}(T^{-1}_\mp(x)\lambda_a T_\mp(x)T^{-1}_\mp(y)\lambda_b T_\mp(y)). \tag{3}$$

Here $D_\pm(x)$ is the Green function fulfilling $\partial D_\pm(x)/\partial x^\mp = \delta^{(2)}(x)$ and the Feynman–Stueckelberg boundary conditions. It is now straightforward to calculate the commutator $[j^a_\pm(x,t), j^b_\pm(y,t)]$ using the BJL definition (Bjorken 1966, Johnson and Low 1966), i.e. taking $x^0 = y^0 \pm \varepsilon$ in (3) and subtracting. Using

$$\frac{\partial D_\pm(x)}{\partial x^\pm} = \delta^{(2)}(x) + \frac{\mathrm{i}}{\pi}\left(\theta(x^0)\frac{1}{(x^\pm - \mathrm{i}\varepsilon)} + \theta(-x^0)\frac{1}{(x^\pm + \mathrm{i}\varepsilon)^2}\right)$$

and taking into account that we are looking for the limit in the sense of distributions over x, y, we get

$$\begin{aligned}
&[j^a_\pm(x,t), j^b_\pm(y,t)] = 2\mathrm{i}f_{abc}j^c_\pm(x,t)\delta(x-y) \pm \frac{\mathrm{i}}{\pi}D^{ab}_{1x}\delta(x-y)\\
&[j^a_\pm(x,t), j^b_\mp(y,t)] = 0.
\end{aligned} \tag{4}$$

The last equation follows from the fact that there is only a local coupling between A_+ and A_- in the effective action $W[A]$. From (4), we obtain further that

$$\begin{aligned}
&[j^a_0(x,t), j^b_0(y,t)] = [j^a_1(x,t), j^b_1(y,t)] = \mathrm{i}f_{abc}j^c_0(x,t)\delta(x-y)\\
&[j^a_0(x,t), j^b_1(y,t)] = \mathrm{i}f_{abc}j^c_1(x,t)\delta(x-y) + \frac{\mathrm{i}}{2\pi}D^{ab}_{1x}\delta(x-y).
\end{aligned} \tag{5}$$

The above results may be confirmed directly by calculating the relevant Feynman graphs. Let us note that the Schwinger terms arising here are of a somewhat unusual form—we have covariant derivatives instead of the usual ones (Gamboa Saravi *et al* 1985b). Equations (5) are covariant and satisfy the Jacobi identities.

We compute now the three-point functions. To this end we introduce the notation:

$$j^k_\pm \equiv j^{a_k}_\pm(x_k) \qquad D^{kl}_\mp \equiv D_\pm(x_k - x_l)$$
$$D'^{kl}_\mp \equiv D'_\pm(x_k - x_l) \qquad M^k_\mp \equiv T^{-1}_\pm(x_k)\lambda_{ak}T_\pm(x_k).$$

Then we have, for example,

$$\langle Tj^i_+ j^k_+ j^m_+\rangle_c \exp(\mathrm{i}W[A])$$
$$= \frac{\mathrm{i}}{\pi} D'^{ik}_+[D^{im}_+ \operatorname{Tr}([M^i_+, M^m_+]M^k_+) + D^{km}_+ \operatorname{Tr}(M^i_+[M^k_+, M^m_+])]. \tag{6}$$

One can check that in spite of the apparent lack of symmetry the above formula is symmetric with respect to any permutation of indices. Using again the BJL definition, we calculate the double commutator. The result is

$$[[j^a_+(x), j^b_+(y)], j^c_+(z)]_{x^0=y^0=z^0} = 2\mathrm{i}f_{abd}[2\mathrm{i}f_{dce}j^e_+(z)\delta(x^1-y^1)\delta(x^1-z^1)$$
$$+\frac{\mathrm{i}}{\pi} D^{dc}_{1x}\delta(x^1-z^1)\delta(x^1-y^1)]. \tag{7}$$

This is consistent with the commutation rules (4). No extra terms related to the singular character of the multiple commutator appear (cf Takahashi 1981, Kololov and Yelkhovsky 1989).

Let us now consider the chiral fermions coupled to the background non-Abelian gauge field

$$S = \int \mathrm{d}^2x\bar{\psi}\gamma^\mu\left[\mathrm{i}\partial_\mu + gA_\mu\left(\frac{1+\gamma_5}{2}\right)\right]\psi$$

with $\gamma_5 = \gamma_0\gamma_1$ we have $\gamma^\mu\gamma_5 = \varepsilon^{\mu\nu}\gamma_\nu$ and the fermion fields couple to A_- only, $S_{\mathrm{int}} = (g/2)\int \mathrm{d}^2x\bar{\psi}\gamma_+A_-\psi$. Therefore the effective action derived along the lines of Makowka and Wanders (1986) is (Falck and Kramer 1987, Manias *et al* 1987)

$$\tilde{W}[A] = K_+[T_+] + \frac{ag^2}{8\pi}\int \mathrm{d}^2x \operatorname{Tr}(A_+A_-)(x).$$

This result is also implicit in the papers of Polyakov and Wiegman (1984) and Rothe (1986). The variable a is a parameter related to regularisation ambiguity (Jackiw and Rajaraman 1985, Manias *et al* 1987, Cabra and Schaposnik 1989).

The current is defined as above

$$\langle\tilde{j}^a_\pm(x)\rangle_A = \frac{-2\mathrm{i}}{g}\frac{\delta}{\delta A^a_\mp(x)}(\exp \mathrm{i}\tilde{W}[A])$$

so that

$$\langle\tilde{j}^a_+(x)\rangle_A = \left(\frac{\mathrm{i}}{2\pi}\operatorname{Tr}(\partial_- T_+(x)T^{-1}_x(x)\lambda_a) + \frac{ag}{8\pi}A^a_+(x)\right)\exp(\mathrm{i}\tilde{W}[A])$$

$$\langle\tilde{j}^a_-(x)\rangle_A = \frac{ag}{8\pi}A^a_-(x)\exp(\mathrm{i}\tilde{W}[A]).$$

One easily checks that the chiral current has an anomalous divergence

$$\langle(D_+\tilde{j}_- + D_-\tilde{j}_+)^a\rangle_A = \frac{g}{4\pi}((a-1)\partial^\mu A^a_\mu + \varepsilon^{\mu\nu}\partial_\mu A^a_\nu)$$

as well as transformation properties

$$\delta\langle \tilde{j}^a_+(x)\rangle_{\text{norm}} = \varepsilon^b(x) f_{bac}\langle \tilde{j}^c_+(x)\rangle + \frac{(a-2)}{8\pi}\partial_+\varepsilon^a(x)$$

$$\delta\langle \tilde{j}^a_-(x)\rangle_{\text{norm}} = \varepsilon^b(x) f_{bac}\langle \tilde{j}^c_-(x)\rangle + \frac{a}{8\pi}\partial_-\varepsilon^a(x)$$

where 'norm' means the in-out matrix element divided by the vacuum persistence amplitude. No choice of the parameter a renders the current covariant.

Applying the same reasoning as above, we arrive at the following commutation rules:

$$\begin{aligned}
&[j^a_0(x,t), j^b_0(y,t)]\\
&\quad = [j^a_1(x,t), j^b_1(y,t)]\\
&\quad = \mathrm{i} f_{abc} j^c_0(x,t)\delta(x-y) + \frac{\mathrm{i}}{4\pi}\delta_{ab}\partial_x\delta(x-y)\\
&\quad\quad - \frac{\mathrm{i}g}{8\pi} f_{abc}[(a-1)A^c_0(x_1 t) + A^c_1(x,t)] \qquad (8)\\
&[j^a_0(x,t), j^b_1(y,t)]\\
&\quad = \mathrm{i} f_{abc} j^c_1(x,t)\delta(x-y) + \frac{\mathrm{i}}{4\pi}\delta_{ab}\partial_x\delta(x-y)\\
&\quad\quad + \frac{\mathrm{i}g}{8\pi} f_{abc}[A^c_0(x,t) - (a+1)A^c_1(x,t)].
\end{aligned}$$

The above results may be compared with those given by Jo (1985). It is easy to check that both the anomalous divergence and commutators coincide if we put $a=0$ (note the opposite chirality is considered here). In fact, the method of calculating the Feynman graphs adopted by Jo gives the effective action depending only on $A_-(A_+)$.

Let us note that although the commutators (8) are obviously not covariant, the Jacobi identity is satisfied.

We may write out the three-point function which is actually given by (6). We used it to calculate the double commutators $[\tilde{j}^a_\mu(x), [\tilde{j}^b_\nu(y), \tilde{j}^c_\sigma(z)]]_{x^0=y^0=z^0}$. Again the result is consistent with the current algebra.

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